1) Lösen Sie die folgenden Integrale durch partielle Integration: $\int u'v = uv - \int uv'$

a) $\int x\ln(x)dx$

Lösung:

$$u = \frac{x^2}{2};\ u' = x;\ v = \ln(x);\ v' = \frac{1}{x};\ \int x\cdot\ln(x)dx = \frac{x^2}{2}\ln(x) - \int\frac{x}{2}dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$$

b) $\int xe^{2x}dx$

Lösung:

$$u = \frac{1}{2}e^{2x};\ u' = e^{2x};\ v = x;\ v' = 1;\ \int x\cdot e^{2x}dx = \frac{1}{2}xe^{2x} - \int\frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

c) $\int x\sqrt{x-3}dx$

Lösung:

$$u = \frac{2}{3}(x-3)^{\frac{3}{2}};\ u' = (x-3)^{\frac{1}{2}};\ v = x;\ v' = 1;$$

$$\int x\cdot\sqrt{x-3}dx = \frac{2}{3}x(x-3)^{\frac{3}{2}} - \frac{2}{3}\int(x-3)^{\frac{3}{2}}dx = \frac{2}{3}x(x-3)^{\frac{3}{2}} - \frac{4}{15}(x-3)^{\frac{5}{2}} + C$$

d) $\int\frac{x}{\sqrt{x+4}}dx$

Lösung:

$$\int\frac{x}{\sqrt{x+4}}dx = \int x(x+4)^{-\frac{1}{2}}dx \quad u = 2(x+4)^{\frac{1}{2}};\ u' = (x+4)^{-\frac{1}{2}};\ v = x;\ v' = 1;$$

$$\int x(x+4)^{-\frac{1}{2}}dx = 2x(x+4)^{\frac{1}{2}} - 2\int(x+4)^{\frac{1}{2}}dx = 2x(x+4)^{\frac{1}{2}} - \frac{4}{3}(x+4)^{\frac{3}{2}} + C$$

2) Lösen Sie die folgenden Integrale durch partielle Integration: $\int u'v = uv - \int uv'$

a) $\int\ln(x)dx$

Lösung:

$$\int\ln(x)dx = \int 1\cdot\ln(x)dx \qquad u = x;\quad u' = 1;\quad v = \ln(x);\quad v' = \frac{1}{x};$$

$$\int\ln(x)dx = x\ln(x) - \int\frac{x}{x}dx = x\ln(x) - \int 1dx = x\ln(x) - x + C$$

b) $\int xe^x dx$

Lösung:

$u = e^x$; $u' = e^x$; $v = x$; $v' = 1$;

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

c) $\int x\cos(x)dx$

Lösung:

$u = \sin(x)$; $u' = \cos(x)$; $v = x$; $v' = 1$;

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + C$$

d) $\int x\sin(3x)dx$

Lösung:

$u = -\frac{1}{3}\cos(3x)$; $u' = \sin(3x)dx$; $v = x$; $v' = 1$;

$$\int x\sin(3x)dx = -\frac{1}{3}x\cos(3x) + \frac{1}{3}\int \cos(3x)dx = -\frac{1}{3}x\cos(3x) + \frac{1}{9}\sin(3x) + C$$

e) $\int e^x \sin(x)dx$

Lösung:

$u = -\cos(x)$; $u' = \sin(x)$; $v = e^x$; $v' = e^x$;

$$\int \sin(x)e^x dx = -\cos(x)e^x + \int \cos(x)e^x dx \qquad u = \sin(x)\ ; u' = \cos(x); \quad v = e^x; \quad v' = e^x;$$

$$\int \sin(x)e^x dx = -\cos(x)e^x + \sin(x)e^x - \int \sin(x)e^x dx \qquad \Big| + \int \sin(x)e^x dx$$

$$2\int \sin(x)e^x dx = -\cos(x)e^x + \sin(x)e^x + C^* \qquad \Big| : 2$$

$$\int \sin(x)e^x dx = -\frac{1}{2}\cos(x)e^x + \frac{1}{2}\sin(x)e^x + C$$

f) $\int x^2 e^x dx$

Lösung:

$u = e^x\ ;\quad u' = e^x;\quad v = x^2;\quad v' = 2x;$

$\int x^2 e^x dx = x^2 e^x - \int 2x \cdot e^x dx \qquad u = e^x\ ;\quad u' = e^x;\quad v = 2x;\quad v' = 2;$

$\int x^2 e^x dx = x^2 e^x - 2xe^x + \int 2 \cdot e^x dx = x^2 e^x - 2xe^x + 2e^x + C$

g) $\int \sin^2(x)dx \qquad Anmerkung: \sin^2(x) + \cos^2(x) = 1$

Lösung:

$\int \sin^2(x)dx = \int \sin(x)\sin(x)dx \qquad u = -\cos(x);\quad u' = \sin(x);\quad v = \sin(x);\quad v' = \cos(x)$

$\int \sin^2(x)dx = -\sin(x)\cos(x) + \int \cos^2(x)dx = -\sin(x)\cos(x) + \int (1 - \sin^2(x))dx$

$\int \sin^2(x)dx = -\sin(x)\cos(x) + \int 1dx - \int \sin^2(x)dx \qquad \left| + \int \sin^2(x)dx \right.$

$2\int \sin^2(x)dx = -\sin(x)\cos(x) + x + C^* \qquad |:2$

$\int \sin^2(x)dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C$

h) $\int x \sin^2(x) dx$

Lösung:

$$u = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x; \quad u' = \sin^2(x); \qquad v = x; \quad v' = 1$$

$$\int x \sin^2(x) dx = -\frac{1}{2}x\sin(x)\cos(x) + \frac{1}{2}x^2 + \int\left(\frac{1}{2}\sin(x)\cos(x) - \frac{1}{2}x\right)dx$$

$$= -\frac{1}{2}x\sin(x)\cos(x) + \frac{1}{2}x^2 + \int\frac{1}{2}\sin(x)\cos(x)dx - \int\frac{1}{2}xdx$$

$$= -\frac{1}{2}x\sin(x)\cos(x) + \frac{1}{2}x^2 - \frac{1}{4}x^2 + \frac{1}{2}\int\sin(x)\cos(x)dx$$

Nebenrechnung :

$$\int\sin(x)\cos(x)dx \qquad u = \sin(x); \qquad \frac{du}{dx} = \cos(x); \quad dx = \frac{du}{\cos(x)}$$

$$= \int u\cos(x)\frac{du}{\cos(x)} = \int u \cdot du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C$$

$$\int x \sin^2(x) dx = -\frac{1}{2}x\sin(x)\cos(x) + \frac{1}{2}x^2 - \frac{1}{4}x^2 + \frac{1}{4}\sin^2(x) + C$$

$$= \frac{1}{4}x^2 - \frac{1}{2}x\sin(x)\cos(x) + \frac{1}{4}\sin^2(x) + C$$