Aufgabe 1

Lösen Sie die folgenden Integrale:

a) $\int (2-3x)^7 \, dx$

Lösung:

$u = 2-3x \qquad \frac{du}{dx} = -3 \qquad dx = \frac{du}{-3}$

$$\int (2-3x)^7 \, dx = \int \frac{u^7}{-3} du = -\frac{1}{24} u^8 + C = -\frac{1}{24}(2-3x)^8 + C$$

b) $\int x \cdot \sin(3-x) dx$

Lösung:

$\int x \cdot \sin(3-x) dx \qquad u = \cos(3-x) \qquad u' = \sin(3-x)$

$\qquad v = x \qquad v' = 1$

$$x\cos(3-x) - \int \cos(3-x) dx = \underline{\underline{x\cos(3-x) + \sin(3-x) + C}}$$

c) $\int x e^{3x} dx$

Lösung:

$u = \frac{1}{3} e^{3x} \qquad u' = e^{3x}$

$v = x \qquad v' = 1;$

$$\int x \cdot e^{3x} dx = \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx = \underline{\underline{\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C}}$$

d) $\int \sin^5(x) \cos(x) dx$

Lösung:

$u = \sin(x) \qquad \frac{du}{dx} = \cos(x) \qquad dx = \frac{du}{\cos(x)}$

$$\int \sin^5(x)\cos(x) dx = \int u^5 \cos(x) \frac{du}{\cos(x)} = \int u^5 du = \frac{1}{6} u^6 + C = \frac{1}{6} \sin^6(x) + C$$

e) $\int \frac{\ln(x)}{x} dx$

Lösung:

$\int \frac{\ln(x)}{x} dx = \int \ln(x) \frac{1}{x} dx \qquad u = \ln(x) \qquad \frac{du}{dx} = \frac{1}{x} \qquad dx = x du$

$$\int \frac{\ln(x)}{x} dx = \int u \frac{1}{x} x \cdot du = \int u \cdot du = \frac{u^2}{2} + C = \frac{1}{2} \ln(x)^2 + C$$

f) $\int x\sqrt{2x-1}dx$

Lösung:

$u = 2x-1 \qquad \frac{du}{dx} = 2 \qquad dx = \frac{1}{2}du \qquad x = \frac{1}{2}(u+1)$

$$\int \frac{1}{2}(u+1)\sqrt{u}\,\frac{1}{2}du = \frac{1}{4}\int (u+1)\sqrt{u}\cdot du = \frac{1}{4}\int\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du = \frac{1}{10}u^{\frac{5}{2}} + \frac{1}{6}u^{\frac{3}{2}} + C$$

$$= \frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + C$$

g) $\int \frac{2x}{x^2-4}dx$

Lösung:

Lösen durch Substitution $\int \frac{f'(x)}{f(x)}dx$

$\int \frac{2x}{x^2-4}dx$ = ln |x² - 4| + C

h) $\int \frac{2x-1}{x^2-x+3}dx$

Lösung:

Lösen durch Substitution $\int \frac{f'(x)}{f(x)}dx$

$\int \frac{2x-1}{x^2-x+3}dx$ = ln|x²-x+3| + C

i) $\int_{-4}^{0}\left(3x^2 - |x+2|\right)dx$

Lösung:

$\int_{-4}^{0}\left(3x^2 - |x+2|\right)dx \qquad x+2=0 \qquad \Rightarrow \qquad x=-2$

$$= \int_{-4}^{-2}\left(3x^2+x+2\right)dx + \int_{-2}^{0}\left(3x^2-x-2\right)dx = \frac{3x^3}{3} + \frac{x^2}{2} + 2x\Big|_{-4}^{-2} + \frac{3x^3}{3} - \frac{x^2}{2} - 2x\Big|_{-2}^{0}$$

$$(-2)^3 + \frac{(-2)^2}{2} + 2(-2) - \left((-4)^3 + \frac{(-4)^2}{2} + 2(-4)\right) + 0 - \left((-2)^3 - \frac{(-2)^2}{2} - 2(-2)\right)$$

$$= -8+2-4+64-8+8+8+2-4 = \underline{\underline{60}}$$

j) $\int x^2 \cdot \sin(x)dx$

Lösung:

$\int x^2 \cdot \sin(x)dx \qquad u = -\cos(x) \qquad u' = \sin(x) \qquad v = x^2 \qquad v' = 2x$

$-x^2\cos(x) + \int 2x\cos(x)dx$ noch mal partiell Ableiten:

$u = \sin(x) \qquad u' = \cos(x) \qquad v = 2x \qquad v' = 2$

$-x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)dx = \underline{\underline{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}}$

Aufgabe 2

a) Gesucht ist die Funktion, die bei x = 1 eine Nullstelle hat und deren Ableitung durch $f(x) = x^3$ dargestellt wird.

Lösung:

$\int x^3 dx = \frac{1}{4}x^4 + C \qquad \frac{1}{4}\cdot 1^4 + C = 0 \qquad \Rightarrow \qquad C = -\frac{1}{4}$

$\underline{\underline{F(x) = \frac{1}{4}x^4 - \frac{1}{4}}}$

b) Bestimmen Sie die Fläche der Figur, die durch folgende Begrenzungen gegeben ist.

$y = \frac{1}{5}x + 2; \quad y = 0; \quad x = 1; \quad x = 5$

Lösung:

$$F_1 = \int_1^5 \left(\frac{1}{5}x + 2\right)dx = \frac{1}{10}x^2 + 2x\Big|_1^5 = \frac{25}{10} + 10 - \left(\frac{1}{10} + 2\right) = 10{,}4$$

Aufgabe 3

Lösen Sie die folgenden Integrale:

a) $\int \frac{t^3}{\sqrt{1+t^4}}dt$

Lösung:

$\int \frac{t^3}{\sqrt{1+t^4}}dt; \qquad u = 1 + t^4 \qquad \frac{du}{dx} = 4t^3 \qquad dx = \frac{du}{4t^3}$

$$= \int \frac{t^3}{\sqrt{u}} \cdot \frac{du}{4t^3} = \frac{1}{4}\int u^{-\frac{1}{2}}du = \frac{1}{4}\cdot 2 \cdot u^{\frac{1}{2}} + C = \frac{1}{2}\sqrt{1+t^4} + C$$

b) $\int_1^3 x \cdot e^{x-3} dx$

Lösung:

$\int_1^3 x \cdot e^{x-3} dx \qquad u = x \qquad u' = 1 \qquad v = e^{x-3} \; v' = e^{x-3}$

$$xe^{x-3}\Big|_1^3 - \int_1^3 e^{x-3} dx = \left(xe^{x-3} - e^{x-3}\right)\Big|_1^3 = 3e^0 - e^0 - e^{-2} + e^{-2} = 3 - 1 = \underline{\underline{2}}$$

c) $\int \frac{x}{\sqrt{x-3}} dx$

Lösung:

$\int \frac{x}{\sqrt{x-3}} dx = \int x(x-3)^{-\frac{1}{2}} dx \qquad u = 2(x-3)^{\frac{1}{2}} \qquad u' = (x-3)^{-\frac{1}{2}}$

$v = x \qquad v' = 1$

$$\int x(x-3)^{-\frac{1}{2}} dx = 2x(x-3)^{\frac{1}{2}} - 2\int (x-3)^{\frac{1}{2}} dx = 2x(x-3)^{\frac{1}{2}} - \frac{4}{3}(x-3)^{\frac{3}{2}} + C$$

d) $\int \arcsin(x) dx$

Lösung:

$\int 1 \cdot \arcsin(x) dx \qquad u = x \qquad u' = 1$

$v = \arcsin(x) \qquad v' = \frac{1}{\sqrt{1-x^2}}$

$x \cdot \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx$ weiter mit Substitution: $u = 1 - x^2 \qquad dx = \frac{du}{-2x}$

$$x \cdot \arcsin(x) + \int \frac{x}{2x \cdot \sqrt{u}} du = x \cdot \arcsin(x) + \frac{1}{2}\int u^{-\frac{1}{2}} du = \underline{\underline{x \cdot \arcsin(x) + \sqrt{1-x^2} + C}}$$

e) $\int x^2 \sqrt{6x^3 - 5} \cdot dx$

Lösung:

$u = 6x^3 - 5 \qquad \frac{du}{dx} = 18x^2 \qquad dx = \frac{1}{18x^2} du;$

$$\int x^2 \sqrt{u} \frac{du}{18x^2} = \frac{1}{18}\int u^{\frac{1}{2}} \cdot du = = \frac{1}{18} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + C = \frac{1}{27}(6x^3 - 5)^{\frac{3}{2}} + C$$

f) $\int \tan(x)dx$

Lösung:

Lösen durch Substitution $\int \frac{f'(x)}{f(x)} dx$

$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)} dx = -\int \frac{-\sin(x)}{\cos(x)} dx$ = -ln |cos(x)| + C

g) $\int \frac{x}{1+x^4} dx$

Lösung:

Substituieren mit u = x^2; $\frac{du}{dx} = 2x; dx = \frac{du}{2x}$;

$\int \frac{x}{1+u^2} \frac{du}{2x} = \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \arctan(u) + C = \frac{1}{2} \arctan(x^2) + C$

h) $\int \frac{\sin(x)}{1+\cos(x)} dx$

Lösung:

Lösen durch Substitution $\int \frac{f'(x)}{f(x)} dx$

$\int \frac{\sin(x)}{1+\cos(x)} dx$ = $-\int \frac{-\sin(x)}{1+\cos(x)} dx$ = $-\ln|1+\cos(x)| + C$

i) $\int \left(1+\sin^2(x)\right)^2 \cos(x)dx$

Lösung:

Substituieren *mit* *t = sin x*

$t = \sin(x); \quad \frac{dt}{dx} = \cos(x); \quad dx = \frac{dt}{\cos(x)}$

$\int \left(1+t^2\right)^2 \cos(x) \frac{dt}{\cos(x)} = \int \left(1+t^2\right)^2 dt = \int \left(1+2t^2+t^4\right) dt = t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C$

$= \sin(x) + \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$

j) $\int x \cdot \sin(x^2)dx$

Lösung:

$\int x \cdot \sin(x^2)dx \qquad u = x^2 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$

$\int x \cdot \sin(x^2)\frac{du}{2x} = \frac{1}{2}\int \sin(u)du = -\frac{1}{2}\cos(u) + C \quad = \underline{\underline{-\frac{1}{2}\cos(x^2) + C}}$

Aufgabe 4

a) Gesucht ist die Stammfunktion zu f(x) = 6x^2 + 4x - 3, die durch den Punkt P(1;6) geht.

Lösung:

$\int(6x^2 + 4x - 3)dx = \frac{6}{3}x^3 + \frac{4}{2}x^2 - 3x + C = 2x^3 + 2x^2 - 3x + C$

$P(1;6) \quad 2 + 2 - 3 + C = 6 \qquad 1 + C = 6 \qquad \Rightarrow \qquad C = 5$

$\underline{\underline{F(x) = 2x^3 + 2x^2 - 3x + 5}}$

b) Bestimmen Sie die Fläche der Figur, die durch folgende Begrenzungen gegeben ist.

$y = x^2 - 5x + 4; \;\; y = 0; \;\; x = 0; \;\; x = 6$

Lösung:
Nullstellen: $x_1 = 1$; $x_2 = 4$; Beide liegen im Intervall => Integral aufspalten:

$I = \int(x^2 - 5x - 4)dx = \frac{x^3}{3} - \frac{5}{2}x^2 - 4x + C;$

$F_1 = I\Big|_0^1 = \frac{1}{3} - \frac{5}{2} + 4 - (0) = \frac{11}{6}; \;\; F_2 = I\Big|_1^4 = \frac{64}{3} - 40 + 16 - \left(\frac{11}{6}\right) = -\frac{9}{2};$

$F_3 = I\Big|_4^6 = 72 - 90 + 24 - \left(\frac{64}{3} - 40 + 16\right) = \frac{26}{3}$

$F = |F_1| + |F_2| + |F_3| = 15$

c) Berechnen Sie den Inhalt der Fläche, die im Intervall [2; 5] von $y = \frac{1}{2}x^2 - x - \frac{3}{2}$ und der x-Achse begrenzt wird.

Lösung:
Nullstellen: $x_1 = -1$; $x_2 = 3$; x_2 liegt im Intervall, darum muss das Integral aufgespaltet werden:

$F_1 = \int_2^3\left(\frac{1}{2}x^2 - x - \frac{3}{2}\right)dx = -\frac{5}{6}; \; F_2 = \int_3^5\left(\frac{1}{2}x^2 - x - \frac{3}{2}\right)dx = \frac{32}{6};$

$F = |F_1| + |F_2| = \frac{5}{6} + \frac{32}{6} = \frac{37}{6} \approx 6{,}17$

Aufgabe 5

Lösen Sie die folgenden Integrale:

a) $\int \frac{x}{\cos^2(x)}dx$

Lösung:

$$\int \frac{x}{\cos^2(x)}dx; \qquad u = x \qquad u' = 1$$

$$v = \tan(x) \qquad v' = \frac{1}{\cos^2(x)}$$

$$\int \frac{x}{\cos^2(x)}dx = x \cdot \tan(x) - \int \tan(x)dx \qquad NR: \int \tan(x)dx = -\int \frac{-\sin(x)}{\cos(x)}dx = -\ln|\cos(x)|$$

$$= x \cdot \tan(x) + \ln|\cos(x)| + C$$

b) $2\int \frac{\cos(x)}{\sin(2x)\cdot\tan(x)}dx$

Lösung:

$$2\int \frac{\cos(x)}{\sin(2x)\cdot\tan(x)}dx; \qquad \sin(2x) = 2\sin(x)\cdot\cos(x); \qquad \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= 2\int \frac{\cos(x)\cdot\cos(x)}{2\sin(x)\cdot\cos(x)\cdot\sin(x)}dx = 4\int \frac{\cos(x)}{\sin^2(x)}dx; \qquad u = \sin(x);\ \frac{du}{dx} = \cos(x);\ dx = \frac{du}{\cos(x)}$$

$$= 4\int \frac{\cos(x)}{u^2}\cdot\frac{du}{\cos(x)} = 4\int u^{-2}du = -4u^{-1} + C = \frac{-4}{\sin(x)} + C$$

c) $\int 3x \cdot e^{\frac{x^2}{3}}dx$

Lösung:

$$\int 3x \cdot e^{\frac{x^2}{3}}dx; \qquad u = \frac{x^2}{3};\ \frac{du}{dx} = \frac{2}{3}x;\ dx = \frac{3du}{2x}$$

$$= \int 3x \cdot e^u \cdot \frac{3du}{2x} = \frac{9}{2}\int e^u du = \frac{9}{2}e^u + C = \frac{9}{2}e^{\frac{x^2}{3}} + C$$

d) $\int \frac{\sin(x)}{e^x} dx$

Lösung:

$$\int \frac{\sin(x)}{e^x} dx = \int \sin(x) e^{-x} dx \qquad u = \sin(x) \qquad u' = \cos(x)$$

$$v = -e^{-x} \qquad v' = e^{-x}$$

$$\int \sin(x) e^{-x} dx = -\sin(x) e^{-x} + \int \cos(x) e^{-x} dx \qquad u = \cos(x) \qquad u' = -\sin(x)$$

$$v = -e^{-x} \qquad v' = e^{-x}$$

$$\int \sin(x) e^{-x} dx = -\sin(x) e^{-x} - \cos(x) e^{-x} - \int \sin(x) e^{-x} dx \qquad \left| + \int \sin(x) e^{-x} dx \right.$$

$$2\int \sin(x) e^{-x} dx = -\sin(x) e^{-x} - \cos(x) e^{-x} \qquad \left| \div 2 \right.$$

$$\int \sin(x) e^{-x} dx = \frac{-\sin(x) e^{-x} - \cos(x) e^{-x}}{2} = \underline{\underline{\frac{-(\sin(x) + \cos(x))}{2e^x}}}$$

e) $\int x \cdot e^{1-3x^2} dx$

Lösung:

$$\int x \cdot e^{1-3x^2} dx; \qquad u = 1 - 3x^2; \ \frac{du}{dx} = -6x; \ dx = \frac{du}{-6x}$$

$$= \int x \cdot e^u \cdot \frac{du}{-6x} = -\frac{1}{6} \int e^u du = -\frac{1}{6} e^u + C = -\frac{1}{6} e^{1-3x^2} + C$$

f) $\int \frac{-x+1}{x^2 - 2x + 2} dx$

Lösung:

$$\int \frac{-x+1}{x^2 - 2x + 2} dx = -\frac{1}{2} \int \frac{2x-2}{x^2 - 2x + 2} dx; \qquad u = x^2 - 2x + 2; \ \frac{du}{dx} = 2x - 2; \ dx = \frac{du}{2x-2}$$

$$= -\frac{1}{2} \int \frac{2x-2}{u} \cdot \frac{du}{2x-2} = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln|u| + C = -\frac{1}{2} \ln\left|x^2 - 2x + 2\right| + C$$

g) $\int \frac{\sin(x)}{\sqrt{2+\cos(x)}}dx$

Lösung:

$$\int \frac{\sin(x)}{\sqrt{2+\cos(x)}}dx; \qquad u = 2+\cos(x);\ \frac{du}{dx} = -\sin(x);\ dx = \frac{du}{-\sin(x)}$$

$$= \int \frac{\sin(x)}{\sqrt{u}} \cdot \frac{du}{-\sin(x)} = -\int u^{-\frac{1}{2}}du = -2u^{\frac{1}{2}} + C = -2\sqrt{2+\cos(x)} + C$$

h) $\int \frac{x}{\sqrt{a^2-x^2}}\,dx$

Lösung:

$$\int \frac{x}{\sqrt{a^2-x^2}}dx; \qquad u = a^2 - x^2;\ \frac{du}{dx} = -2x;\ dx = \frac{du}{-2x}$$

$$= \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x} = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2} \cdot 2 \cdot u^{\frac{1}{2}} + C = -\sqrt{a^2-x^2} + C$$